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Abstract The problem of state estimation for heat non-homogeneous equations under distributed in space measurements is considered. An interval observer is designed, described by Partial Differential Equations (PDEs), for uncertain distributed parameter systems without application of finite-element approximations. Conditions of boundedness of solutions of interval observer with non-zero boundary conditions and measurement noise are proposed. The results are illustrated by numerical experiments with an academic example.

*Keywords:* Interval observers, Heat equation, Distributed parameter system

## 1. INTRODUCTION

Due to various technical (complexity of implementation) or economic (price of solution) issues, an explicit measurement of state vector of a dynamical system may be impossible. This is especially the case, for example, in distributed parameter systems, where the system state is a function of the space and time, and only pointwise and discrete measurements are realizable by a sensor. Consequently, the system state in these cases has to be reconstructed using estimation algorithms Meurer et al. (2005); Fossen and Nijmeijer (1999); Besanon (2007). The most popular approaches in this domain include Luenberger observer and Kalman filter for deterministic and stochastic settings, respectively, which are developed for linear time-invariant models, that is the case where the existing theory disposes of many solutions. For nonlinear dynamical systems, state estimation algorithms are often based on a partial similarity of the plant models to linear ones, or representations in various canonical forms are widely used.

Various physical phenomena, can be formalized in terms of PDEs (*e.g.* sound, heat, electrostatics, electrodynamics, fluid flow, elasticity, or quantum mechanics), whose distributed nature introduces additional level of complexity in design. That is why control and estimation of PDEs is a very popular direction of research nowadays Bredies et al. (2013); Smyshlyaev and Krstic (2010). Frequently, for design of a state estimator or control, the finite-dimensional approximation approach is used Alvarez and Stephanopoulos (1982); Dochain (2000); Vande Wouver and Zeitz (2002); Hagen and Mezic (2003), then the control or estimation problems are addressed in the framework of finite-dimensional systems using well-known tools. Analysis and design in the original distributed coordinates are more complicated, but also attract attention of many researchers Smyshlyaev and Krstic (2010); Hidayat et al. (2011); Ahmed-Ali et al. (2015).

Inline with the model complexity, the system uncertainty represents another difficulty for synthesis of an estimator or controller. The uncertainty may consist in unknown parameters or/and external disturbances. Appearance of uncertainty fails the design of a conventional estimator, converging to the ideal value of the state. In this case an interval estimation becomes more attainable: an observer can be constructed such that using input-output information it evaluates the set of admissible values (interval) for the state at each instant of time. The interval width is proportional to the size of the model uncertainty (it has to be minimized by tuning the observer parameters). There are several approaches to design interval/set-membership estimators Jaulin (2002); Kieffer and Walter (2004); Olivier and Gouzé (2004). This work is devoted to interval observers, which form a subclass of set-membership estimators and whose design is based on the monotone systems theory Olivier and Gouzé (2004); Moisan et al. (2009); Raïssi et al. (2010, 2012); Efimov et al. (2012). The idea of interval observer design has been proposed rather recently in Gouzé et al. (2000), but it has already received numerous extensions for various classes of dynamical models. Interval observers for systems described by PDEs have been proposed in Perez and Moura (2015); Kharkovskaya

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et al. (2016). The finite-dimensional approximation approach was applied in Kharkovskaya et al. (2016) using the discretization error estimates from Wheeler (1973). In Perez and Moura (2015) the sensitivity function interval estimates are used for an interval observer design.

In the present paper an extension of this approach for estimation of systems described by PDEs is discussed. Using the conditions of positivity of solutions of parabolic PDEs presented in Nguyn and Coron (2016), an interval observer is constructed governed by PDE, whose estimation error dynamics (also distributed) is positive. The stability analysis from Fridman and Blighovsky (2012) is also extended to the considered scenario with non-zero measurement noise and boundary conditions.

The outline of this paper is as follows. After preliminaries in Section 2, and introduction of distributed parameter system properties in Section 3, the interval observer design is given in Section 4. The results of numerical experiments with a simple parabolic equation are presented in Section 5.

## 2. PRELIMINARIES: POSITIVITY OF FINITE-DIMENSIONAL SYSTEMS

The real numbers are denoted by $\mathbb{R}$, $\mathbb{R}_+ = \{\tau \in \mathbb{R} : \tau \geq 0\}$. Euclidean norm for a vector $x \in \mathbb{R}^n$ will be denoted as $|x|$.

### *2.1 Interval relations*

For two vectors $x_1, x_2 \in \mathbb{R}^n$ or matrices $A_1, A_2 \in \mathbb{R}^{n\times n}$, the relations $x_1 \leq x_2$ and $A_1 \leq A_2$ are understood elementwise. The relation $P \prec 0$ ($P \succ 0$) means that the matrix $P \in \mathbb{R}^{n\times n}$ is negative (positive) definite. Given a matrix $A \in \mathbb{R}^{m\times n}$, define $A^+ = \max\{0, A\}$, $A^- = A^+ - A$ (similarly for vectors) and denote the matrix of absolute values of all elements by $|A| = A^+ + A^-$.

*Lemma 1.* Efimov et al. (2012) Let $x \in \mathbb{R}^n$ be a vector variable, $\underline{x} \leq x \leq \overline{x}$ for some $\underline{x}, \overline{x} \in \mathbb{R}^n$.

(1) If $A \in \mathbb{R}^{m\times n}$ is a constant matrix, then

$$A^+\underline{x} - A^-\overline{x} \leq Ax \leq A^+\overline{x} - A^-\underline{x}. \tag{1}$$

(2) If $A \in \mathbb{R}^{m\times n}$ is a matrix variable and $\underline{A} \leq A \leq \overline{A}$ for some $\underline{A}, \overline{A} \in \mathbb{R}^{m\times n}$, then

$$\begin{aligned} \underline{A}^+\underline{x}^+ - \overline{A}^+\underline{x}^- - \underline{A}^-\overline{x}^+ + \overline{A}^-\overline{x}^- &\leq Ax \\ \leq \overline{A}^+\overline{x}^+ - \underline{A}^+\overline{x}^- - \overline{A}^-\underline{x}^+ + \underline{A}^-\underline{x}^-&. \end{aligned} \tag{2}$$

Furthermore, if $-\overline{A} = \underline{A} \leq 0 \leq \overline{A}$, then the inequality (2) can be simplified: $-\overline{A}(\overline{x}^+ + \underline{x}^-) \leq Ax \leq \overline{A}(\overline{x}^+ + \underline{x}^-)$.

### *2.2 Nonnegative continuous-time linear systems*

A matrix $A \in \mathbb{R}^{n\times n}$ is called Hurwitz if all its eigenvalues have negative real parts, it is called Metzler if all its elements outside the main diagonal are nonnegative (exponential $e^A$ of a Metzler matrix $A$ is a nonnegative matrix Farina and Rinaldi (2000)). Any solution of the linear system

$$\dot{x}(t) = Ax(t) + B\omega(t),\ \omega : \mathbb{R}_+ \to \mathbb{R}_+^q,$$

with $x(t) \in \mathbb{R}^n$ and a Metzler matrix $A \in \mathbb{R}^{n\times n}$, is elementwise nonnegative for all $t \geq 0$ provided that $x(0) \geq 0$ and $B \in \mathbb{R}_+^{n\times q}$ Farina and Rinaldi (2000); Smith (1995). The output solution

$$y(t) = Cx(t) + D\omega(t),$$

where $y(t) \in \mathbb{R}^p$, is nonnegative if $C \in \mathbb{R}_+^{p\times n}$ and $D \in \mathbb{R}_+^{p\times q}$. Such dynamical systems are called cooperative (monotone) or nonnegative if only initial conditions in $\mathbb{R}_+^n$ are considered Farina and Rinaldi (2000); Smith (1995).

## 3. POSITIVITY AND STABILITY OF HEAT EQUATION

In this section the basic facts on PDE and positivity of solutions of distributed parameter systems are given.

### *3.1 Preliminaries*

If $X$ is a normed space with norm $||\cdot||_X$, $\Omega \subset \mathbb{R}^n$ for some $n \geq 1$ and $\phi : \Omega \to X$, define

$$||\phi||^2_{L^2(\Omega,X)} = \int_\Omega ||\phi(s)||^2_X ds,$$
$$||\phi||_{L^\infty(\Omega,X)} = ess\sup_{s\in\Omega} ||\phi(s)||_X.$$

By $L^\infty(\Omega, X)$ and $L^2(\Omega, X)$ denote the set of functions $\Omega \to X$ with the properties $||\cdot||_{L^\infty(\Omega,X)} < +\infty$ and $||\cdot||_{L^2(\Omega,X)} < +\infty$, respectively. Denote $I = [0, \ell]$ for some $\ell > 0$, let $C^k(I, X)$ be the set of functions having continuous derivatives through order $k \geq 0$ on $I$. For any $q > 0$ and an interval $I' \subseteq I$ define $W^{q,\infty}(I', \mathbb{R})$ as a subset of functions $y \in C^{q-1}(I', \mathbb{R})$ with an absolutely continuous $y^{(q-1)}$ and bounded $y^{(q)}$ on $I'$, $||y||_{W^{q,\infty}} = \sum_{i=0}^q ||y^{(r)}||_{L^\infty(I',\mathbb{R})}$. Denote by $H^q(I, \mathbb{R})$ with $q \geq 0$ the Sobolev space of functions with derivatives through order $q$ in $L^2(I, \mathbb{R})$, the dual space of $H^q(I, \mathbb{R})$ will be denoted as $H^{-q}(I, \mathbb{R})$.

For two functions $\phi_1, \phi_2 : I \to \mathbb{R}$ their relation $\phi_1 \leq \phi_2$ has to be understood as $\phi_1(x) \leq \phi_2(x)$ for almost all $x \in I$, the inner product is defined in a standard way:

$$(\phi_1, \phi_2) = \int_0^\ell \phi_1(x)\phi_2(x)dx.$$

For $\phi \in \mathbb{R}$ define two operators $\phi^+$ and $\phi^-$ as follows:

$$\phi^+ = \max\{0, \phi\},\ \phi^- = \phi^+ - \phi.$$

*Lemma 2.* Kharkovskaya et al. (2016) Let $s, \underline{s}, \overline{s} : I \to \mathbb{R}$ admit the relations $\underline{s} \leq s \leq \overline{s}$, then for any $\phi : I \to \mathbb{R}$ we have

$$(\underline{s}, \phi^+) - (\overline{s}, \phi^-) \leq (s, \phi) \leq (\overline{s}, \phi^+) - (\underline{s}, \phi^-).$$

### *3.2 Heat equation*

Consider the following PDE with associated boundary conditions:

$$\begin{aligned} \frac{\partial z(x,t)}{\partial t} &= L[x, z(x,t)] + r(x,t) \quad \forall (x,t) \in I \times \mathcal{T}, \\ z(x,t_0) &= z_0(x) \quad \forall x \in I, \\ z(0,t) &= \alpha(t),\ z(\ell,t) = \beta(t) \quad \forall t \in \mathcal{T}, \end{aligned} \tag{3}$$

where $I = [0, \ell]$ with $0 < \ell < +\infty$, $\mathcal{T} = [t_0, t_0 + T)$ for $t_0 \in \mathbb{R}$ and $T > 0$,

$$L(x,z) = \frac{\partial}{\partial x}\left(a(x)\frac{\partial z}{\partial x}\right) + q(x)z,$$

$a \in H^2(I,\mathbb{R})$, $q \in H^1(I,\mathbb{R})$ and there exist $a_{\min}, a_{\max} \in \mathbb{R}_+$ such that

$$0 < a_{\min} \le a(x) \le a_{\max} \quad \forall x \in I;$$

the boundary conditions $\alpha, \beta \in L^2(\mathcal{T},\mathbb{R})$ and the external input $r \in L^2(I \times \mathcal{T},\mathbb{R})$; the initial conditions $z_0 \in H^{-1}(I,\mathbb{R})$.

Under these restrictions the Cauchy problem (3) is well posed in $C^0(\mathcal{T}, H^{-1}(I,\mathbb{R}))$, and there exist a unique solution $z(x,t)$ and a constant $\rho$ independent of $\alpha$, $\beta$, $r$ and $z_0$ such that Nguyn and Coron (2016):

$$\begin{aligned}||z||_{C^0(\mathcal{T},H^{-1}(I,\mathbb{R}))} &\le \rho(||\alpha||_{L^2(\mathcal{T},\mathbb{R})} \\ &+||\beta||_{L^2(\mathcal{T},\mathbb{R})} + ||r||_{L^2(I\times\mathcal{T},\mathbb{R})} \\ &+||z_0||_{H^{-1}(I,\mathbb{R})}).\end{aligned}$$

*Proposition 3.* Let $\alpha, \beta \in H^1(\mathcal{T},\mathbb{R})$ and $a_{\min}\frac{\pi^2}{\ell^2} = q_{\max} + \chi$, where $\chi > 0$ and $q_{\max} = \sup_{x\in I} q(x)$, then for solutions of (3) the following estimate is satisfied for all $t \in \mathcal{T}$:

$$\begin{aligned}\frac{1}{2}\int_0^\ell z^2(x,t)dx &\le e^{-\chi(t-t_0)}\int_0^\ell w_0^2(x)dx + \chi^{-2}\int_0^\ell \tilde{r}^2(x,t)dx \\ &+\frac{\ell}{2}[\alpha^2(t)+\beta^2(t)],\end{aligned}$$

where $w_0(x) = z_0(x) - \delta(x,t_0)$, $\delta(x,t) = \alpha(t) + \frac{x}{\ell}(\beta(t) - \alpha(t))$ and

$$\begin{aligned}\tilde{r}(x,t) &= r(x,t) + \frac{1}{\ell}\frac{\partial a(x)}{\partial x}(\beta(t) - \alpha(t)) \\ &+q(x)\delta(x,t) - \delta_t(x,t).\end{aligned}$$

All proofs are excluded due to space limitations. Another variant of stability proof for Proposition 3 can be found in Alcaraz-Gonzlez et al. (2005). Consequently, Proposition 3 fixes the conditions under which the distributed parameter system (3) possesses the input-to-state stability property Dashkovskiy et al. (2011); Dashkovskiy and Mironchenko (2013), where boundary conditions $\alpha$, $\beta$ influence the external disturbance $r$ and the initial conditions as well. The main restriction of that proposition is

$$a_{\min}\frac{\pi^2}{\ell^2} > q_{\max},$$

that can be easily validated for a sufficiently small $\ell$.

Note that after a straightforward calculus the estimate from Proposition 3 can be rewritten as follows for all $t \in \mathcal{T}$:

$$\begin{aligned}||z(\cdot,t)||^2_{L^2(I,\mathbb{R})} &\le 4e^{-\chi(t-t_0)}[||z_0||^2_{L^2(I,\mathbb{R})} + \varrho(t_0)] \\ &+8\chi^{-2}||r(\cdot,t)||^2_{L^2(I,\mathbb{R})} + \gamma(t),\end{aligned}$$

where $\varrho(t) = \frac{\ell}{2}[\alpha^2(t) + \beta^2(t)]$ (weighted norm of the boundary conditions), $\varrho'(t) = \frac{\ell}{2}[\dot{\alpha}^2(t) + \dot{\beta}^2(t)]$ (weighted norm of derivative of the boundary conditions) and $\gamma(t) = 8\chi^{-2}\varrho'(t) + 2(1 + 4\frac{q^2_{\max}}{\chi^2} + 16\frac{\partial a^2_{\max}}{\chi^2\ell^2})\varrho(t)$ are all bounded functions of time $t \in \mathcal{T}$, $\partial a_{\max} = \sup_{x\in I}\frac{\partial a(x)}{\partial x}$.

### 3.3 Positivity of solutions

In general, the solution $z(\cdot,t)$ of (3) takes its values in $\mathbb{R}$ and it can change sign with $(x,t) \in I \times \mathcal{T}$.

*Definition 4.* The distributed parameter system (3) is called nonnegative (positive) on the interval $\mathcal{T}$ if for

$$\alpha(t) \ge 0,\ ,\beta(t) \ge 0,\ r(x,t) \ge 0 \quad \forall (x,t) \in I \times \mathcal{T}$$

the implication $z_0(x) \ge 0 \Rightarrow z(x,t) \ge 0$ ($z_0(x) > 0 \Rightarrow z(x,t) > 0$) holds for all $(x,t) \in I \times \mathcal{T}$ and for all $z_0 \in H^{-1}(I,\mathbb{R})$.

A well-known example of a nonnegative system is homogeneous heat equation defined over $x \in (-\infty,+\infty)$:

$$\begin{aligned}\frac{\partial z(x,t)}{\partial t} &= a\frac{\partial^2 z(x,t)}{\partial^2 x} + r(x,t) \quad \forall (x,t) \in \mathbb{R}\times\mathcal{T}, \qquad (4)\\ z(x,0) &= z_0(x) \quad \forall x \in \mathbb{R},\end{aligned}$$

where $a > 0$ and $z_0 : \mathbb{R} \to \mathbb{R}_+$, whose solution can be calculated analytically using Green's function (fundamental solution or heat kernel) Thomée (2006):

$$\begin{aligned}z(x,t) &= \frac{1}{2\sqrt{\pi a t}}\int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4at}} z_0(y)dy \\ &+\int_0^t\int_{-\infty}^{+\infty}\frac{e^{-\frac{(x-y)^2}{4a(t-s)}}}{2\sqrt{\pi a(t-s)}} r(y,s)dy\ ds.\end{aligned}$$

It is straightforward to verify that for nonnegative $z_0$ and $r$ the expression in the right-hand side stays nonnegative for all $(x,t) \in \mathbb{R}\times(0,+\infty)$. This conclusion is valid for the case $x \in \mathbb{R}$, and if $x \in I$, even in (4) with $r(x,t) = 0$ for all $(x,t) \in I \times \mathcal{T}$, and with the boundary condition

$$0 = z(0,t) = z(\ell,t) \quad \forall t \in \mathcal{T} \qquad (5)$$

the heat equation admits the solution in the form:

$$\begin{aligned}z(x,t) &= \sum_{n=1}^{+\infty} D_n \sin(\frac{n\pi x}{\ell})e^{-a\frac{n^2\pi^2}{\ell^2}t}, \\ D_n &= \frac{2}{\ell}\int_0^\ell z_0(x)\sin(\frac{n\pi x}{\ell})dx,\end{aligned}$$

whose positivity is less trivial to establish.

For this reason, using Maximum principle Friedman (1964) the following general result has been established in Nguyn and Coron (2016):

*Proposition 5.* Let $\alpha, \beta \in L^2(\mathcal{T},\mathbb{R}_+)$, $r \in L^2(I \times \mathcal{T},\mathbb{R}_+)$ and $z_0 \in H^{-1}(I,\mathbb{R}_+)$, then

$$z(x,t) \ge 0 \quad \forall (x,t) \in I \times \mathcal{T},$$

*i.e.* (3) is nonnegative on the interval $\mathcal{T}$.

Therefore, if boundary and initial conditions, and external inputs, take only nonnegative values, then the solutions of (3) possess the same property.

## 4. INTERVAL OBSERVER DESIGN FOR HEAT EQUATION

Consider (3) with some uncertain boundary conditions $\alpha, \beta \in L^2(\mathcal{T},\mathbb{R})$, an uncertain external input $r \in L^2(I \times \mathcal{T},\mathbb{R})$ and initial conditions $z_0 \in H^{-1}(I,\mathbb{R})$, and assume that the state $z(x,t)$ is available for measurements in certain points $0 \le x_1^m < x_2^m < \cdots < x_p^m \le \ell$:

$$y_i(t) = z(x_i^m,t) + \nu_i(t),\ i = 1,\ldots,p, \qquad (6)$$

where $y(t) = [y_1(t),\ldots,y_p(t)]^T \in \mathbb{R}^p$ is the measured output signal, $\nu(t) = [\nu_1(t),\ldots,\nu_p(t)] \in \mathbb{R}^p$ and $\nu \in$

$L^\infty(\mathbb{R}_+,\mathbb{R}^p)$ is the measurement noise. Design of a conventional observer under similar conditions has been studied in Fridman and Blighovsky (2012); Schaum et al. (2014).

The goal of the work consists in design of interval observers for the distributed parameter system (3), (6). For this purpose we need the following hypothesis.

*Assumption 1.* Let $\underline{z}_0 \le z_0 \le \overline{z}_0$ for some known $\underline{z}_0, \overline{z}_0 \in H^{-1}(I,\mathbb{R})$, let also functions $\underline{\alpha}, \overline{\alpha}, \underline{\beta}, \overline{\beta} \in L^2(\mathcal{T},\mathbb{R})$, $\underline{r}, \overline{r} \in L^2(I\times\mathcal{T},\mathbb{R})$ and a constant $\nu_0 > 0$ be given such that for all $(x,t) \in I\times\mathcal{T}$:

$$\underline{\alpha}(t) \le \alpha(t) \le \overline{\alpha}(t),\ \underline{\beta}(t) \le \beta(t) \le \overline{\beta}(t),$$
$$\underline{r}(x,t) \le r(x,t) \le \overline{r}(x,t),\ |\nu(t)| \le \nu_0.$$

Thus, by Assumption 1 five intervals, $[\underline{\alpha}(t), \overline{\alpha}(t)]$, $[\underline{\beta}(t), \overline{\beta}(t)]$, $[\underline{z}_0, \overline{z}_0]$, $[\underline{r}(x,t), \overline{r}(x,t)]$ and $[-\nu_0, \nu_0]$, determine for all $(x,t) \in I\times\mathcal{T}$ in (3), (6) uncertainty of the values for $\alpha(t)$, $\beta(t)$, $z_0$, $r(x,t)$ and $\nu(t)$, respectively.

The simplest interval observer for (3) under the introduced assumptions is as follows for $i = 0,1,\dots,p$:

$$\begin{aligned}
\frac{\partial \overline{z}(x,t)}{\partial t} &= L[x,\overline{z}(x,t)] + \overline{r}(x,t) \quad \forall (x,t) \in I_i\times\mathcal{T},\\
\overline{z}(x,t_0) &= \overline{z}_0(x) \quad \forall x \in I_i,\\
\overline{z}(x_i^m,t) &= \overline{Z}_i(t),\ \overline{z}(x_{i+1}^m,t) = \overline{Z}_{i+1}(t) \quad \forall t \in \mathcal{T}; \qquad (7)\\
\frac{\partial \underline{z}(x,t)}{\partial t} &= L[x,\underline{z}(x,t)] + \underline{r}(x,t) \quad \forall (x,t) \in I_i\times\mathcal{T},\\
\underline{z}(x,t_0) &= \underline{z}_0(x) \quad \forall x \in I_i,\\
\underline{z}(x_i^m,t) &= \underline{Z}_i(t),\ \underline{z}(x_{i+1}^m,t) = \underline{Z}_{i+1}(t) \quad \forall t \in \mathcal{T},
\end{aligned}$$

where $\overline{z} \in C^0(\mathcal{T}, H^{-1}(I,\mathbb{R}))$ and $\underline{z} \in C^0(\mathcal{T}, H^{-1}(I,\mathbb{R}))$ are upper and lower estimates of the solution $z(x,t)$; $I_i = [x_i^m, x_{i+1}^m]$ with $x_0^m = 0$ and $x_{p+1}^m = \ell$; the upper and lower estimates for the boundary conditions are

$$\overline{Z}(t) = [\overline{\alpha}(t),\ y_1(t)+\nu_0,\dots,y_p(t)+\nu_0,\ \overline{\beta}(t)]^T,$$
$$\underline{Z}(t) = [\underline{\alpha}(t),\ y_1(t)-\nu_0,\dots,y_p(t)-\nu_0,\ \underline{\beta}(t)]^T.$$

Therefore, the domain $I$ of the solution of (3) is divided on $p+1$ subdomains with appropriate boundary conditions.

*Assumption 2.* Let $\alpha,\beta \in H^1(\mathcal{T},\mathbb{R})$ and $\nu \in H^1(\mathcal{T},\mathbb{R}^p)$.

Note that the subsystems for $\overline{z}(x,t)$ and $\underline{z}(x,t)$ in the PDE (7) are isolated, and each of them is of the same class as (3) under Assumption 2, then the Cauchy problem (7) is well posed in $C^0(\mathcal{T}, H^{-1}(I,\mathbb{R}))$, and there exists unique solutions $\overline{z}(x,t)$ and $\underline{z}(x,t)$ Nguyn and Coron (2016). In addition Nguyn and Coron (2016):

$$\begin{aligned}
||\overline{z}||_{C^0(\mathcal{T},H^{-1}(I,\mathbb{R}))} &\le \rho(||\overline{Z}||_{L^2(\mathcal{T},\mathbb{R})}\\
&+||\overline{r}||_{L^2(I\times\mathcal{T},\mathbb{R})} + ||\overline{z}_0||_{H^{-1}(I,\mathbb{R})}),\\
||\underline{z}||_{C^0(\mathcal{T},H^{-1}(I,\mathbb{R}))} &\le \rho(||\underline{Z}||_{L^2(\mathcal{T},\mathbb{R})}\\
&+||\underline{r}||_{L^2(I\times\mathcal{T},\mathbb{R})} + ||\underline{z}_0||_{H^{-1}(I,\mathbb{R})}).
\end{aligned}$$

The upper and lower interval estimation errors for (3) and (7) can be introduced as follows:

$$\overline{e}(x,t) = \overline{z}(x,t) - z(x,t),\ \underline{e}(x,t) = z(x,t) - \underline{z}(x,t),$$

whose dynamics take the form for $i = 0,1,\dots,p$:

$$\begin{aligned}
\frac{\partial \overline{e}(x,t)}{\partial t} &= L[x,\overline{e}(x,t)] + \overline{r}(x,t)\\
&\quad -r(x,t) \quad \forall (x,t) \in I_i\times\mathcal{T},\\
\overline{e}(x,t_0) &= \overline{z}_0(x) - z_0(x) \quad \forall x \in I_i,\\
\overline{e}(x_i^m,t) &= \overline{Z}_i(t) - z(x_i^m,t) \quad \forall t \in \mathcal{T},\\
\overline{e}(x_{i+1}^m,t) &= \overline{Z}_{i+1}(t) - z(x_{i+1}^m,t) \quad \forall t \in \mathcal{T};\\
\frac{\partial \underline{e}(x,t)}{\partial t} &= L[x,\underline{e}(x,t)] + r(x,t) \qquad (8)\\
&\quad -\underline{r}(x,t) \quad \forall (x,t) \in I_i\times\mathcal{T},\\
\underline{e}(x,t_0) &= z_0(x) - \underline{z}_0(x) \quad \forall x \in I_i,\\
\underline{e}(x_i^m,t) &= z(x_i^m,t) - \underline{Z}_i(t) \quad \forall t \in \mathcal{T},\\
\underline{e}(x_{i+1}^m,t) &= z(x_{i+1}^m,t) - \underline{Z}_{i+1}(t) \quad \forall t \in \mathcal{T}.
\end{aligned}$$

*Theorem 6.* Let assumptions 1 and 2 be satisfied, then in (3), (7):

$$\underline{z}(x,t) \le z(x,t) \le \overline{z}(x,t) \quad \forall (x,t) \in I\times\mathcal{T}.$$

In addition, if

$$\Delta x^m < 2\pi\sqrt{\frac{a_{\min}}{q_{\max}}}, \qquad (9)$$

where $\Delta x^m = \max_{0\le i\le p} x_{i+1}^m - x_i^m$, then for all $t \in \mathcal{T}$:

$$\begin{aligned}
||\overline{z}(\cdot,t) - z(\cdot,t)||^2_{L^2(I,\mathbb{R})} &\le 4e^{-\chi(t-t_0)}[||\overline{z}_0 - z_0||^2_{L^2(I,\mathbb{R})} + \overline{\varrho}(t_0)]\\
&\quad +8\chi^{-2}||\overline{r}(\cdot,t) - r(\cdot,t)||^2_{L^2(I,\mathbb{R})} + \overline{\gamma}(t),\\
||z(\cdot,t) - \underline{z}(\cdot,t)||^2_{L^2(I,\mathbb{R})} &\le 4e^{-\chi(t-t_0)}[||z_0 - \underline{z}_0||^2_{L^2(I,\mathbb{R})} + \underline{\varrho}(t_0)]\\
&\quad +8\chi^{-2}||r(\cdot,t) - \underline{r}(\cdot,t)||^2_{L^2(I,\mathbb{R})} + \underline{\gamma}(t),
\end{aligned}$$

where

$$\overline{\varrho}(t) = \ell||\overline{Z}(t) - Z(t)||^2,\ \overline{\varrho}'(t) = \ell||\dot{\overline{Z}}(t) - \dot{Z}(t)||^2,$$
$$\overline{\gamma}(t) = 8\chi^{-2}\overline{\varrho}'(t) + 2(1 + 4\frac{q_{\max}^2}{\chi^2} + 16\frac{\partial a_{\max}^2}{\chi^2\ell^2})\overline{\varrho}(t),$$
$$\underline{\varrho}(t) = \ell||Z(t) - \underline{Z}(t)||^2,\ \underline{\varrho}'(t) = \ell||\dot{Z}(t) - \dot{\underline{Z}}(t)||^2,$$
$$\underline{\gamma}(t) = 8\chi^{-2}\underline{\varrho}'(t) + 2(1 + 4\frac{q_{\max}^2}{\chi^2} + 16\frac{\partial a_{\max}^2}{\chi^2\ell^2})\underline{\varrho}(t)$$

and

$$Z(t) = [\alpha(t),\ y^T(t) - \nu^T(t),\ \beta(t)]^T.$$

It is well-known fact that the system (8) can be unstable if the function $q$ takes sufficiently big values Curtain and Zwart (1995). In Fridman and Blighovsky (2012) it has been proven, for $\alpha(t) = \beta(t) = 0$ and $\nu(t) = 0$, that the observer (7) is asymptotically stable if the difference $\Delta x^m$ is sufficiently small *(i.e.* there are sufficient quantity of sensors uniformly distributed in $I$). The presented Theorem 6 ensures positiveness of the interval estimation errors and boundedness of the interval estimates $\overline{z}$ and $\underline{z}$ in the presence of non-zero boundary conditions $\alpha(t)$, $\beta(t)$ and measurement noise $\nu(t)$.

## 5. EXAMPLE

Consider an academic example of (3) for

$$a(x) = 1 + \frac{3}{4}\sin(2\pi x),\ q(x) = \frac{1}{2}\cos(\pi x),$$
$$r(x,t) = \sin(\pi x)[\cos(2t) + \epsilon(t)],\ |\epsilon(t)| \le 1,$$

with $T = 10$ and $\ell = 1$, then $\epsilon$ is an uncertain part of the input $r$ (for simulation $\epsilon(t) = \cos(10t)$), and

$\underline{r}(x,t) = \sin(\pi x)[\cos(2t) - 1]$, $\overline{r}(x,t) = \sin(\pi x)[\cos(2t) + 1]$.

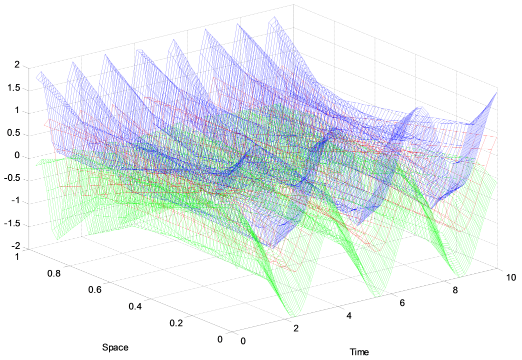


Figure 1. The results of interval estimation for $N = 10$

The uncertainty of initial conditions is given by the interval

$$\underline{z}_0(x) = z_0(x) - 1,\ \overline{z}_0(x) = z_0(x) + 1,$$

where $z_0(x) = \sin(\pi x)$, and for boundary initial conditions

$$\underline{\alpha}(t) = \sin(2t) - 1,\ \overline{\alpha}(t) = \sin(2t) + 1,$$
$$\underline{\beta}(t) = \sin(5t) - 1,\ \overline{\beta}(t) = \sin(5t) + 1,$$

where $\alpha(t) = \sin(2t)$ and $\beta(t) = \sin(5t)$. Let $p = 3$ with $x_1^m = 0.2$, $x_2^m = 0.5$, $x_3^m = 0.8$, and

$$\nu(t) = 0.1[\sin(20t)\ \sin(15t)\ \cos(25t)]^{\mathrm{T}},$$

then $\nu_0 = 0.173$. In this case $\Delta x^m = 0.3$, $a_{\min} = 0.25$, $q_{\max} = 0.5$ and the restriction (9) is verified, therefore all conditions of Theorem 6 are satisfied.

For calculation of scalar product in space and for simulation of the discretized PDE in time, the explicit and implicit Euler methods have been used, respectively, with the step 0.01. The results of interval estimation are shown in Fig. 1, where the red surface corresponds to $z(x, t)$, while green and blue ones represent $\underline{z}(x, t)$ and $\overline{z}(x, t)$, respectively (20 and 40 points are used for plotting in space and in time).

*Remark 7.* Note that since for calculation of solutions the finite-element discretization/approximation methods are use, then their error of approximation has to be taken into account in the final estimates in order to ensure the desired interval inclusion property for all $x \in I$ and $t \in \mathcal{T}$, see Kharkovskaya et al. (2016) where the result from Wheeler (1973) was applied for an evaluation of this error.

## 6. CONCLUSION

Taking a parabolic PDE with Dirichlet boundary conditions, a method of design of interval observers is proposed, which is not based on a finite-element approximation. The conditions of positivity of solutions of hyperbolic PDEs proposed in Nguyn and Coron (2016) are taken into account in the design. The efficiency of the proposed interval observer is demonstrated through numerical experiments.

For future developments, the proposed interval observer can be used for control design of an uncertain PDE system in the spirit of Efimov et al. (2013), and a more complex uncertainty of PDE equation can also be incorporated in the design procedure.

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